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WELL-POSEDNESS OF WEAK SOLUTION FOR NONLINEAR PARABOLIC PROBLEM VIA OPTIMIZATION METHOD

L. Hmidouch[1*], A. Jamea[2], M. Laghdir[1]

[1]Laboratoire LAROSERI, Faculté des Sciences, Université Chouaib Doukkali, El Jadida, Morocco
[2]CRMEF Casablanca-Settat, S.P. El Jadida, El Jadida, Morocco

Abstract. We prove in this paper some existence and uniqueness results of weak solutions for some nonlinear degenerate parabolic problem with Dirichlet type boundary condition. The data are assumed to be L^∞.

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Corresponding author: Hmidouch, Lhoucine, Laboratoire LAROSERI, Faculté des Sciences, Université Chouaib Doukkali, El Jadida, Morocco, e-mail: *hmidouchlhoucine@gmail.com*
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1 Introduction

Much research has been conducted in the last few decades on nonlinear degenerate parabolic problems, detected in mathematics, biology, chemistry and physics Nachaoui et al. (2021); Rasheed et al. (2021); Yakub et al. (2021). For example, the flow through a porous medium in a turbulent regime is described by

$$\frac{\partial \theta}{\partial t} + div\, v = 0,$$

and Darcy's law

$$v = -K(\theta) \text{ grad } \phi(\theta),$$

where $\theta(x,t)$ is the volumetric moisture content, $k(\theta)$ is the hydraulic conductivity, and the total potential ϕ is given by

$$\phi(\theta) = \psi(\theta) + z,$$

with $\psi(\theta)$ the hydrostatic potential and z the gravitational potential. In turbulent regimes, the flow rate is different from that which can be predicted by the Darcy's law, and so several authors have proposed a nonlinear relation between v and $K(\theta)$ grad ϕ.

$$|v|^{q-2}v = -K(\theta)\text{grad } \phi(\theta), \quad q > 2.$$

If e denotes the unit vector in the vertical direction, we obtain

$$\frac{\partial \theta}{\partial t} - \text{div} \mid \nabla\varphi(\theta) - K(\theta)e \mid^{p-2} (\nabla\varphi(\theta) - K(\theta)e) = 0,$$

where

$$\varphi(\theta) = \int_0^\theta K(s)\phi'(s)ds, \quad p = \frac{q}{q-1}.$$

As we know there is no general theory that concerns the solvability of nonlinear parabolic differential equations, however, many researchers have investigated the existence and the stability of different types of solutions of nonlinear degenerate parabolic problems due to their applications. For example Blanchard & Porrettal (2001), have studied the existence and the stability of a renormalized solution of a nonlinear parabolic equation with a local quadratic term with respect to the gradient and measure initial data. In El Hachimi et al. (2010) have explained the existence of entropy solutions of the nonlinear parabolic problem using a time discretization of the continuous problem by the Euler forward scheme. In Xu & Zho (2005), Xu and Zho established the existence and uniqueness of weak solutions for the initial-boundary value problem of a fourth-order nonlinear parabolic equation. The literature about the results of nonlinear parabolic equations is immense and it is very difficult to have a complete picture, we refer the readers to see for example Abassi et al. (2008); Abassi & El Hachimi (2007); Diaz & Thelin (1994); Blanchard & Redwane (1998).

Motivated by the physical models, we have carried out this study of the weak solutions of a nonlinear degenerate parabolic problem with Dirichlet type boundary conditions

$$\begin{cases} \dfrac{\partial u}{\partial t} - div\ \Phi\left(\nabla u - \theta\left(u\right)\right) + \alpha\left(u\right) = f \quad in \quad Q :=]0;T[\times\Omega, \\ u = 0 \qquad\qquad\qquad\qquad\qquad\qquad on \quad \Gamma :=]0;T[\times\partial\Omega, \\ u\left(.,0\right) = u_0 \qquad\qquad\qquad\qquad\qquad\qquad in \qquad \Omega, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ is a bounded open set, T is a fixed positive number,∇u is the gradient of u, $f \in L^\infty(Q)$, α and θ are the functions defined on $\mathbb{R}$ and satisfied suitable assumptions, and

$$\Phi(\mu) = |\mu|^{p-2}\mu, \qquad \forall \mu \in \mathbb{R}^N.$$

Problems like (1) appear in a variety of different settings (see Bermudez et al. (1984), Diaz & Herrero (1981), Van Duijn & Hilhorst (1987)). Besides that, it is worth mentioning that the problem for the elliptic case with Neumann conditions has been studied in Abassi et al. (2008). For the Dirichlet type boundary conditions, many particular cases have been treated in many works. For example, M. Tsutsumi (1972) investigated, for $\theta = 0$ and $\alpha(u) = u^{1+s}$ where $s \geq 0$, the existence and nonexistence of solutions for (1) using Dirichlet boundary conditions. In Bhuvaneswari et al. (2012) the authors established the existence of weak solutions for the degenerate p-Laplacian parabolic problem using a semi-discretization process. Recently Cianchi & Mazýa (2020), have proved, for $\theta = 0$ and $\alpha(u) = 0$, that the equation (1) has a unique approximable solution.

To prove the existence and uniqueness of the weak solutions, we use a variational method with the following semi-discretization equation

$$\begin{cases} \dfrac{u_k - u_{k-1}}{h} - div\ \Phi(\nabla u_k - \theta(u_k)) + \alpha\left(u_k\right) = [f]_h\left(\left(k-1\right)h\right) \ in \quad \Omega, \\ u_k|_{\partial\Omega} = 0, \qquad\qquad\qquad\qquad\qquad\qquad k = 1, \ldots, n, \end{cases} \tag{2}$$

where $h > 0$, n is a positive integer such that $h = \frac{T}{n}$, and

$$[f]_h\left(x,t\right) = \frac{1}{h}\int_t^{t+h} f\left(x,\tau\right) \mathrm{d}\tau.$$

As we know, many of these partial differential equations can be derived in a variational way, i.e. via minimization of an 'energy' functional. Recently, there has been increasing interest from applied analysts in applying the models and techniques from variational methods and partial differential equations to tackle problems in data science, see Trillos & Murray (2017); Elmoataz et al. (2017); Bernal et al. (2017). Moreover, the variational method and the semi-discretization

process have been used by several authors in different PDEs, see for example Abassi & El Hachimi (2007); Benzekri & El Hachimi (2003); Eden et al. (1990); Chen (2017); Xu & Zho (2005); Zhang & Zhou (2010).
The paper is planned in the following way. In section 2, we state some preliminary results and tools, which are needed to establish our existence result. In section 3, we prove the existence of weak solutions of semi-discrete problem (2) using variational method. In section 4, we establish the existence of weak solutions of the problem (1) using semi-discretization process.

2 Preliminaries and notations

We begin by recalling some relevant definitions and results from calculus and measure theory that we will need throughout this article. Let Ω be a smooth bounded domain in $\mathbb{R}^N$, $diam(\Omega)$ represents its diameter of Ω, the norm in $L^p(\Omega)$ is denoted by $\|\|_{L^p(\Omega)}$, $\|\|_{W^{1,p}(\Omega)}$ denotes the norm in the Sobolev space $W^{1,p}(\Omega)$, $C_c^\infty(\Omega)$ denotes the space of all functions with compact support in Ω with continuous derivatives of arbitrary order, and $W_0^{1,p}(\Omega)$ represents the closure $C_c^\infty(\Omega)$ in $W^{1,p}(\Omega)$. We recall that the dual space of the Sobolev spaces $W_0^{1,p}(\Omega)$ is equivalent to $W^{-1,p'}(\Omega)$, where p' is the conjugate of p i.e., $p' = \frac{p}{p-1}$. For a Banach space X and $a < b$, $L^p(a;b;X)$ is the space of measurable functions $u : [a;b] \mapsto$ X such that

$$\|u\|_{L^p(a,b;X)} := \left(\int_a^b \|u(t)\|_X^p \mathrm{d}t\right)^{1/p} < \infty.$$

Throughout this paper, we will use the following Poincare inequality (see Lieberman (1991), Lemma 2.2), there exists a positive constant $r = diam(\Omega)$ such that

$$\|v\|_{L^p(\Omega)} \leq \frac{r}{2}\|\nabla v\|_{L^p(\Omega)}. \tag{3}$$

We recall the following useful lemmas.

Lemma 1. *(Abassi et al. (2008))* $\forall \mu, \nu \in \mathbb{R}^N$ *and* $1 < p < \infty$

$$\frac{1}{p}|\mu|^p - \frac{1}{p}|\nu|^p \leq |\mu|^{p-2}\mu(\mu - \nu).$$

Lemma 2. *(Boccado & Croce (2014)) Let* f_n *be a sequence of function and* f *be a function in* $L^p(\Omega)$, $p > 1$. *Assume that*
(1) f_n *is bounded in* $L^p(\Omega)$.
(2) $f_n \to f$ *a.e. in* Ω.
Then $f_n \to f$ *strong in* L^q, *for every* $q \in [1, p)$ *and weakly in* $L^p(\Omega)$.

Lemma 3. *For* $\mu, \nu \in \mathbb{R}^N$ *and* $1 < p < \infty$, *we have*

$$\left(|\mu|^{p-2}\mu - |\nu|^{p-2}\nu\right) \cdot (\mu - \nu >) \geq 0.$$

Lemma 4. *For* $a \geq 0,\ b \geq 0$ *and* $1 \leq p < \infty$, *we have*

$$(a+b)^p \leq 2^{p-1}\left(a^p + b^p\right).$$

3 Existence and uniqueness of solutions of semi-discrete problem

To analyze the solutions of the semi-discrete problem (2), it suffices to determine existence of weak solutions of the following elliptic problem

$$\begin{cases} \dfrac{u-u_0}{h} - div\ \Phi\left(\nabla u - \theta\left(u\right)\right) + \alpha(u) = [f]_h\left(0\right) & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{4}$$

under the given assumptions:

(H_1) α is a non decreasing continuous real function defined on $\mathbb{R}$ such that $\alpha(0)=0$ and there exists a positive constant λ_1 such that $|\alpha(x)| \leq \lambda_1|x|$ for all $x \in \mathbb{R}$, $0<\lambda_1<\frac{1}{2}$.

(H_2) θ is a continuous function from $\mathbb{R}$ to $\mathbb{R}^N$ such that $\theta(0)=0$ and $|\theta(x)-\theta(y)| \leq \lambda_2|x-y|$ for all $(x,y) \in \mathbb{R}^2$ and λ_2 is a positive constant such that $0<\lambda_2<\frac{1}{diam(\Omega)}(\frac{1}{2})^{\frac{1}{p}}$.

(H_3) $f \in L^{\infty}(Q)$.

Before we proceed to the proof of (4), we introduce the notion of a weak solution to (4).

Definition 1. *A function $u \in W_0^{1,p}(\Omega) \cap L^2(\Omega)$ is called a weak solution of the problem (4) if, for any $\varphi \in W_0^{1,p}(\Omega) \cap L^2(\Omega)$, we have*

$$\int_\Omega \frac{u-u_0}{h}\varphi \mathrm{d}x + \int_\Omega \Phi(\nabla u - \theta(u))\nabla\varphi \mathrm{d}x + \int_\Omega \alpha(u)\varphi \mathrm{d}x = \int_\Omega [f]_h(0)\varphi \mathrm{d}x.$$

In order to establish existence of solutions to the problem (4), we introduce the variation problem

$$\min\left\{J(u)/u \in W_0^{1,p}(\Omega) \cap L^2(\Omega)\right\},$$

for

$$J(u) := \frac{1}{2h}\int_\Omega (u-u_0)^2 \mathrm{d}x + \frac{1}{h}\int_\Omega \beta(u)\mathrm{d}x + Z(u), \tag{5}$$

where β and Z are functions defined by

$$\beta(t) := h\int_0^t \alpha(s)\mathrm{d}s,$$
$$Z(u) := \frac{1}{p}\int_\Omega |\nabla u - \theta(u)|^p \mathrm{d}x - \int_\Omega |f|_h(0)u\mathrm{d}x.$$

Solving the varational problem enables us to ensure the existence of weak solutions of the elliptic problem (4). To do that, first we need the following result.

Lemma 5. *Provided that (H_2) holds, there exists $C>0$ such that*

$$Z(u) \geq \frac{1}{p2^p}\int_\Omega |\nabla u|^p \mathrm{d}x - C\left\|[f]_h(0)\right\|_{L^{p'}(\Omega)}^{p'}, \quad \forall u \in W_0^{1,p}(\Omega) \cap L^2(\Omega).$$

Proof. Let $u \in W_0^{1,p}(\Omega) \cap L^2(\Omega)$. By applying Lemma 4 and the hypothesis (H_2), we get

$$\begin{aligned}
\frac{1}{p2^{p-1}}|\nabla u|^p &= \frac{1}{p}|\nabla u - \theta(u) + \theta(u)|^p \\
&\leq \frac{1}{p}|\nabla u - \theta(u)|^p + \frac{1}{p}|\theta(u)|^p \\
&\leq \frac{1}{p}|\nabla u - \theta(u)|^p + \frac{\lambda_2^p}{p}|u|^p \\
&\leq \frac{1}{p}|\nabla u - \theta(u)|^p + \frac{1}{2p(diam(\Omega))^p}|u|^p.
\end{aligned}$$

On the other hand, we have by (3)

$$\frac{1}{p2^{p-1}}\int_{\Omega}|\nabla u|^{p}\ \mathrm{d}x\leq\frac{1}{p}\int_{\Omega}|\nabla u-\theta(u)|^{p}\mathrm{d}x+\frac{1}{p2^{p+1}}\int_{\Omega}|\nabla u|^{p}\mathrm{d}x$$

i.e.,

$$\frac{3}{p2^{p+1}}\int_{\Omega}|\nabla u|^{p}\mathrm{d}x\leq\frac{1}{p}\int_{\Omega}|\nabla u-\theta\left(u\right)|^{p}\mathrm{d}x. \tag{6}$$

This implies that

$$Z(u)\geq\frac{3}{p2^{p+1}}\int_{\Omega}|\nabla u|^{p}\ \mathrm{d}x+\int_{\Omega}|f|_{h}\left(0\right)u\ \mathrm{d}x.$$

We obtain by Hölder's, (3) and Young's inequalities

$$\begin{aligned}\left|\int_{\Omega}[f]_{h}(0)u\mathrm{d}x\right|&\leq\left\|[f]_{h}(0)\right\|_{L^{p'}(\Omega)}\left\|u\right\|_{L^{p}(\Omega)}\\&\leq\frac{r}{2}\left\|[f]_{h}(0)\right\|_{L^{p'}(\Omega)}\left\|\nabla u\right\|_{L^{p}(\Omega)}\\&\leq\epsilon\|\nabla u\|_{L^{p}(\Omega)}^{p}+C(\epsilon)\left\|[f]_{h}(0)\right\|_{L^{p'}(\Omega)}^{p'},\end{aligned} \tag{7}$$

where ϵ is a small positive number. Then, from (6) and (7), we have

$$Z(u)\geq\left(\frac{3}{p2^{p+1}}-\epsilon\right)\int_{\Omega}|\nabla u|^{p}\mathrm{d}x-C(\epsilon)\left\|[f]_{h}(0)\right\|_{L^{p'}(\Omega)}^{p'}.$$

Taking $\epsilon=\frac{1}{p2^{p+1}}$, we get

$$Z(u)\geq\frac{1}{p2^{p}}\int_{\Omega}|\nabla u|^{p}\mathrm{d}x-C\left\|[f]_{h}(0)\right\|_{L^{p'}(\Omega)}^{p'}$$

which proves the statement. □

We are now able to determine existence of minimizer of the functional J.

Proposition 1. *Provided that (H_1), and (H_2) hold, then the functional J has a minimizer*

$$u\in W_{0}^{1,p}\left(\Omega\right)\cap L^{2}(\Omega).$$

Proof. Let $u\in W_{0}^{1,p}\left(\Omega\right)\cap L^{2}(\Omega)$. By lemma 5, we have

$$J(u)\geq\frac{1}{2h}\int_{\Omega}(u-u_{0})^{2}\mathrm{d}x+\frac{1}{h}\int_{\Omega}\beta(u)\mathrm{d}x+\frac{1}{p2^{p}}\int_{\Omega}|\nabla u|^{p}\mathrm{d}x-C\left\|[f]_{h}(0)\right\|_{L^{p'}(\Omega)}^{p'}.$$

On the one hand, by applying the hypothesis (H_1), we obtain

$$\begin{aligned}J(u)&\geq\frac{1}{2h}\int_{\Omega}(u-u_{0})^{2}\mathrm{d}x-\frac{\lambda_{1}}{2h}\int_{\Omega}u^{2}\mathrm{d}x-C\|[f]_{h}(0)\|_{L^{p'}(\Omega)}^{p'}\\&\geq\frac{1}{2h}\int_{\Omega}(u-u_{0})^{2}\mathrm{d}x-\frac{1}{4h}\int_{\Omega}u^{2}\mathrm{d}x-C\|[f]_{h}(0)\|_{L^{p'}(\Omega)}^{p'}\\&=\frac{1}{4h}\int_{\Omega}\left(u-2u_{0}\right)^{2}\mathrm{d}x-\frac{1}{2h}\int_{\Omega}u_{0}^{2}\ \mathrm{d}x-C\|[f]_{h}(0)\|_{L^{p'}(\Omega)}^{p'}.\end{aligned} \tag{8}$$

Hence

$$J(u)\geq-\frac{1}{2h}\int_{\Omega}u_{0}^{2}\mathrm{d}x-C\|[f]_{h}(0)\|_{L^{p'}(\Omega)}^{p'}.$$

This implies that

$$-\frac{1}{2h}\|u_0\|^2_{L^2(\Omega)} - C\|[f]_h(0)\|^{p'}_{L^{p'}(\Omega)} \le \inf_{u\in W_0^{1,p}(\Omega)\cap L^2(\Omega)} J(u) \le \frac{1}{2h}\|u_0\|^2_{L^2(\Omega)}.$$

Then, we can find a minimizing sequence $\{u_m\} \subset W_0^{1,p}(\Omega)\cap L^2(\Omega)$ such that

$$J(u_m) \le J(u_0) + 1, \tag{9}$$

and

$$\lim_{m\to\infty} J(u_m) = \inf_{u\in W_0^{1,p}(\Omega)\cap L^2(\Omega)} J(u).$$

From (8) and (9), we get

$$\begin{aligned}\frac{1}{4h}\int_\Omega (u_m-2u_0)^2\mathrm{d}x - \frac{1}{2h}\int_\Omega u_0^2\mathrm{d}x + \frac{1}{p2^p}\int_\Omega |\nabla u_m|^p\mathrm{d}x \\ \le C\|[f]_h(0)\|^{p'}_{L^{p'}(\Omega)} + \frac{1}{2h}\|u_0\|^2_{L^2(\Omega)} + 1.\end{aligned} \tag{10}$$

Since we have

$$\frac{1}{2}u_m^2 - 4u_0^2 \le (u_m-2u_0)^2,$$

we obtain

$$\frac{1}{8h}\|u_m\|^2_{L^2(\Omega)} + \frac{1}{p2^p}\|u_m\|^p_{W_0^{1,p}(\Omega)} \le C\|[f]_h(0)\|^{p'}_{L^{p'}(\Omega)} + \frac{2}{h}\|u_0\|^2_{L^2(\Omega)} + 1.$$

Hence, the above inequality shows that u_m is bounded in $W_0^{1,p}(\Omega)\cap L^2(\Omega)$. Since $W_0^{1,p}(\Omega)\cap L^2(\Omega)$ is reflexive, then we can find subsequence denotes u_m, and a function $u \in$ such that $u_m \rightharpoonup u$ in $W_0^{1,p}(\Omega)\cap L^2(\Omega)$.
Therefore

$$u_m \rightharpoonup u \text{ weakly in } L^p(\Omega), L^2(\Omega), \tag{11}$$

$$\nabla u_m \rightharpoonup \nabla u \text{ weakly in } L^p(\Omega), \tag{12}$$

$$u_m \to u \text{ a.e in } \Omega. \tag{13}$$

Next, we need to show that

$$\liminf_{m\to\infty} J(u_m) \ge J(u).$$

Since $u_m \to u$ a.e in Ω, by Fatou's Lemma, we have

$$\liminf_{m\to\infty}\frac{1}{2h}\int_\Omega (u_m-u_0)^2\,\mathrm{d}x \ge \frac{1}{2h}\int_\Omega (u-u_0)^2\,\mathrm{d}x. \tag{14}$$

So by hypothesis (H_2), we get

$$|\theta(u_m)| \le \lambda_2|u_m|, \tag{15}$$

witch implies that θ is bounded in $L^p(\Omega)$. Moreover, by the continuity of θ we have

$$\lim_{m\to\infty}\theta(u_m) = \theta(u).$$

By applying Lemma 2, we obtain $\theta(u_m) \rightharpoonup \theta(u)$ weakly in $L^p(\Omega)$.
We also have, from (12), $\nabla u_m - \theta(u_m) \rightharpoonup \nabla u - \theta(u)$ weakly in $L^p(\Omega)$,
which gives

$$\liminf_{m\to\infty}\frac{1}{p}\int_\Omega |\nabla u_m - \theta(u_m)|^p\mathrm{d}x \ge \frac{1}{p}\int_\Omega |\nabla u - \theta(u)|^p\mathrm{d}x. \tag{16}$$

By hypothesis (H_1), we have

$$|\beta\left(u_m\right)| \leq \frac{h\lambda_1}{2}|u_m|^2,$$

which implies that β is bounded in $L^2(\Omega)$. Since β is continuous, then

$$\lim_{m\to\infty}\beta\left(u_m\right) = \beta\left(u\right).$$

Thus, by Lemma(2), we get

$$\lim_{m\to\infty}\frac{1}{h}\int_\Omega \beta(u_m)\mathrm{d}x = \frac{1}{h}\int_\Omega \beta(u)\mathrm{d}x. \tag{17}$$

And by (1), we obtain

$$\lim_{m\to\infty}\int_\Omega [f]_h(0)u_m\mathrm{d}x = \int_\Omega [f]_h(0)u\mathrm{d}x. \tag{18}$$

Combining (14), (16), (17) and (18), we deduce

$$\liminf_{m\to\infty} J(u_m) \geq J(u).$$

Therefore u is a minimizer of the functional J in $W_0^{1,p}(\Omega)\cap L^2\left(\Omega\right)$. □

The following is a crucial result to establish existence of solutions to (4).

Lemma 6. *For any u, v in $W_0^{1,p}(\Omega)\cap L^2\left(\Omega\right)$, we have*

(i) $\displaystyle\lim_{t\to0}\int_\Omega \frac{\beta(u+tv)-\beta(u)}{ht}\mathrm{d}x = \int_\Omega \alpha(u)v\mathrm{d}x.$

(ii) $\displaystyle\lim_{t\to0}\int_\Omega \frac{|\nabla u+t\nabla v-\theta(u+tv)|^p-|\nabla u-\theta(u)|^p}{pt}\mathrm{d}x = \int_\Omega \Phi(\nabla u-\theta(u)\cdot\nabla v\mathrm{d}x.$

Proof. (i) Consider, for $t\in]0,1[$,

$$\begin{aligned} G:[0,1] &\longrightarrow \mathbb{R}\\ \mu &\longmapsto \frac{\beta\left(u+t\mu v\right)-\beta\left(u\right)}{ht}.\end{aligned}$$

The function G is continuous on [0,1], and differentiable on]0,1[. By the Mean Value Theorem, there exists $\gamma\in]0,1[$ such that

$$G\left(1\right)-G\left(0\right) = G'\left(\gamma\right).$$

Then

$$\frac{\beta\left(u+tv\right)-\beta\left(u\right)}{ht} = \frac{1}{h}\beta'\left(u+t\gamma v\right)v = \alpha\left(u+t\gamma v\right)v.$$

Since $\gamma, t\in]0,1[$, it implies that

$$|\alpha\left(u+t\gamma v\right)v| \leq \lambda_1|u||v|+\lambda_1|v|^2 \leq |u||v|+|v|^2.$$

On the other hand

$$\lim_{t\to0}\frac{\beta(u+tv)-\beta\left(u\right)}{ht} = \alpha(u)v.$$

Hence, by the Dominated Convergence Theorem, we obtain

$$\lim_{t\to0}\int_\Omega \frac{\beta(u+tv)-\beta(u)}{ht}\mathrm{d}x = \int_\Omega \alpha(u)v\mathrm{d}x.$$

(ii) To show that

$$\lim_{t\to 0}\int_\Omega \frac{|\nabla u+t\nabla v-\theta(u+tv)|^p-|\nabla u-\theta(u)|^p}{pt}\mathrm{d}x=\int_\Omega \Phi(\nabla u-\theta(u)\cdot\nabla v\mathrm{d}x,$$

we define, for $t\in]0,1[$,

$$G_t(u,v)=\nabla u+t\nabla v-\theta(u+tv),\quad F_t(u,v)=\nabla u+t\nabla v-\theta(u),$$

and let M be a function defined as

$$\begin{aligned} M:[0,1]&\longrightarrow\mathbb{R}\\ \mu&\longmapsto\frac{|F_{t\mu}(u,v)|^p-|F_0(u,v|^p}{pt}. \end{aligned}$$

The function M is continuous on [0,1] and differentiable on]0,1[, then again by the Mean Value Theorem, there exists $\gamma\in]0,1[$ such that

$$M\left(1\right))-M\left(0\right))=M'\left(\gamma\right).$$

Thus

$$\frac{|F_t(u,v|^p-|F_0(u,v|^p}{pt}=|F_{t\gamma}(u,v|^{p-2}(F_{t\gamma}(u,v))\cdot\nabla v.$$

Since $\gamma,\ t\in[0,1]$, then by Young's inequalities and Lemma 4, we get

$$\begin{aligned} |F_{t\gamma}(u,v|^{p-2}\left(F_{t\gamma}(u,v)\right)\cdot\nabla v &\leq |F_{t\gamma}(u,v)|^{p-1}|\nabla v|\\ &\leq \frac{|F_{t\gamma}(u,v)|^p}{p'}+\frac{|\nabla v|^p}{p}\\ &\leq 2^{p-1}\left(\frac{|\nabla u-\theta(u)|^p+|t\gamma\nabla v|^p}{p'}\right)+\frac{|\nabla v|^p}{p}\\ &\leq 2^{p-1}\left(\frac{|\nabla u-\theta(u)|^p+|\nabla v|^p}{p'}\right)+\frac{|\nabla v|^p}{p}. \end{aligned}$$

On the other hand

$$\lim_{t\to 0}\frac{|\nabla u+t\nabla v-\theta(u)|^p-|\nabla u-\theta(u)|^p}{pt}=\Phi\left(\nabla u-\theta(u)\cdot\nabla v.\right.$$

Hence, by the Dominated Convergence Theorem, we obtain

$$\lim_{t\to 0}\int_\Omega \frac{|\nabla u+t\nabla v-\theta(u)|^p-|\nabla u|^p}{pt}\mathrm{d}x=\int_\Omega \Phi\left(\nabla u-\theta(u)\cdot\nabla v\mathrm{d}x.\right. \tag{19}$$

To finish the proof of (ii), it suffices to show that

$$\lim_{t\to 0}\int_\Omega \frac{|\nabla u+t\nabla v-\theta(u+tv)|^p-|\nabla u+t\nabla v-\theta(u)|^p}{pt}\mathrm{d}x=0.$$

Using Lemma 1 and by applying Hölder's and Young's inequalities, we get

$$\begin{aligned} \int_\Omega \frac{|G_t(u,v)|^p-|F_t(u,v))|^p}{pt}\mathrm{d}x &\leq \int_\Omega \frac{|G_t(u,v)|^{p-2}(G_t(u,v)\cdot(G_t(u,v)-F_t(u,v))}{t}\\ &\leq \int_\Omega \frac{|G_t(u,v)|^{p-1}|\theta(u+tv)-\theta(u)|}{t}\mathrm{d}x\\ &\leq \frac{\|G_t(u,v)\|_{L^p(\Omega)}^{\frac{p}{p'}}\|\theta(u+tv)-\theta(u)\|_{L^p(\Omega)}}{t}\\ &\leq 3\epsilon\frac{\|G_t(u,v)\|_{L^p(\Omega)}^{\frac{4p}{3p'}}}{4t}+\frac{\|\theta(u+tv)-\theta(u)\|_{L^p(\Omega)}^4}{4\epsilon t}\\ &\leq 3\epsilon\frac{\|G_t(u,v)\|_{L^p(\Omega)}^{\frac{4p}{3p'}}}{4t}+\frac{\lambda_2^4\|tv\|_{L^p(\Omega)}^4}{4\epsilon t}. \end{aligned}$$

Choosing $\epsilon = t^2$, we have

$$\int_\Omega \frac{|G_t(u,v)|^p - |F_t(u,v)|^p}{t} \leq \frac{3t\|G_t(u,v)\|_{L^p(\Omega)}^{\frac{4p}{3p'}}}{4} + \frac{\lambda_2^4 t\|v\|_{L^p(\Omega)}^4}{4}. \tag{20}$$

In the same manner, we prove that

$$\int_\Omega \frac{|G_t(u,v)|^p - |F_t(u,v)|^p}{t} \geq \frac{-3t\|F_t(u,v)\|_{L^p(\Omega)}^{\frac{4p}{3p'}}}{4} - \frac{\lambda_2^4 t\|v\|_{L^p(\Omega)}^4}{4}. \tag{21}$$

Then we obtain from (20) and (21), as $t \to 0$,

$$\lim_{h\to 0} \int_\Omega \frac{|G_t(u,v)|^p - |F_t(u,v)|^p}{t} \mathrm{d}x = 0. \tag{22}$$

Therefore

$$\lim_{t\to 0} \int_\Omega \frac{|\nabla u + t\nabla v - \theta(u+tv)|^p - |\nabla u + t\nabla v - \theta(u)|^p}{pt} \mathrm{d}x = 0. \tag{23}$$

Combining (19) and (23), we obtain

$$\lim_{t\to 0} \int_\Omega \frac{|\nabla u + t\nabla v - \theta(u+tv)|^p - |\nabla u - \theta(u)|^p}{pt} \mathrm{d}x = \int_\Omega \Phi(\nabla u - \theta(u) \cdot \nabla v \mathrm{d}x.$$

□

Now we are ready to prove the existence of weak solutions of the problem (4).

Theorem 1. *Assume that $u_0 \in L^2(\Omega)$, and the hypotheses $(H_1), (H_2), (H_3)$ hold. Then the problem (4) has a unique weak solution.*

Proof. Since u is a minimizer of the functional J in $W_0^{1,p}(\Omega) \cap L^2(\Omega)$, then for any $v \in V$ we have

$$\begin{aligned} 0 &\leq \frac{J(u+tv) - J(u)}{t} \\ &= \int_\Omega \frac{u-u_0}{h} v \mathrm{d}x + t\int_\Omega \frac{v^2}{h}\, \mathrm{d}x + \int_\Omega \frac{\beta(u+t\gamma v) - \beta(u)}{ht} \mathrm{d}x \\ &\quad + \int_\Omega \frac{|\nabla u + t\nabla v - \theta(u+tv)|^p - |\nabla u - \theta(u)|^p}{pt} \mathrm{d}x - \int_\Omega [f]_h(0) v \mathrm{d}x. \end{aligned} \tag{24}$$

By letting $t \to 0$ and using Lemma 6, we deduce from (24)

$$0 \leq \int_\Omega \frac{1}{h}(u-u_0) v \mathrm{d}x + \int_\Omega \alpha(u) v \mathrm{d}x + \int_\Omega \Phi(\nabla u - \theta(u) \cdot \nabla v \mathrm{d}x - \int_\Omega [f]_h(0) v \mathrm{d}x.$$

Thus

$$\int_\Omega \frac{1}{h}(u-u_0) v \mathrm{d}x + \int_\Omega \alpha(u) v \mathrm{d}x + \int_\Omega \Phi(\nabla u - \theta(u) \cdot \nabla v \mathrm{d}x - \int_\Omega [f]_h(0) v \mathrm{d}x = 0,$$

which completes our proof of existence of weak solutions of the problem (4).
The proof of the uniqueness of the solution of the equation (4) is similar to the proof of Theorem 3.3 in El Hachimi et al. (2010) and so it is omitted. □

The following is an immediate consequence of Theorem 1.

Corollary 1. *Assume that $u_0 \in L^2(\Omega)$, and the hypotheses (H_1), (H_2) and (H_3) hold. Then for each $k = 1, \ldots, n$, the problem (2) has a unique weak solution $u_k \in W_0^{1,p}(\Omega) \cap L^2(\Omega)$.*

Proof. For $k = 1$, by Theorem 4, there exists a weak solution $u_1 \in W_0^{1,p}(\Omega) \cap L^2(\Omega)$. By induction, we deduce in the same manner that for $k = 2, \ldots, n$, the problem (2) has a unique weak solution $u_k \in W_0^{1,p}(\Omega) \cap L^2(\Omega)$. □

4 Main result

Before we sate the main result of this section, we introduce the notion of weak solutions to the problem (1).

Definition 2. *We say that* $u \in L^p\left(0,T;W_0^{1,p}(\Omega)\right) \cap L^\infty\left(0,T;L^2(\Omega)\right) \cap C(0,T;L^2(\Omega))$ *such that* $\frac{\partial u}{\partial t} \in L^{p'}\left(0,T;W^{-1,p'}(\Omega)\right)$ *is a weak solution for the parabolic problem (1) if and only if*

$$\int_0^T \int_\Omega \frac{\partial u}{\partial t}\varphi \mathrm{d}x\mathrm{d}t + \int_0^T \int_\Omega \Phi(\nabla u - \theta(u))\nabla\varphi \mathrm{d}x\mathrm{d}t + \int_0^T \int_\Omega \alpha(u)\varphi \mathrm{d}x\mathrm{d}t = \int_0^T \int_\Omega f\varphi \mathrm{d}x\mathrm{d}t,$$

$\forall \varphi \in L^p\left(0,T;W_0^{1,p}(\Omega)\right) \cap L^\infty\left(0,T;L^2(\Omega)\right).$

Now the following is the main result of this paper.

Theorem 2. *Assume that* $u_0 \in L^2(\Omega)$*, and the hypotheses* $(H_1),(H_2),(H_3)$ *hold. Then the problem (1) has a unique weak solution.*

In order to prove the above theorem, we establish the following lemmas.

Definition 3. *Fix* n *a positive integer and let* $h = \frac{T}{n}$*. For* $k = 1,\cdots,n$*, let* u_k *be a solution of (2). The approximate solution* u_h *of (2) is defined by*

$$u_h(x,t) = \begin{cases} u_0(x), & t = 0 \\ u_1(x), & 0 < t \le h \\ \cdots, & \cdots \\ u_j(x), & (j-1)h < t \le jh \\ \cdots, & \cdots \\ u_n(x), & (n-1)h < t \le nh = T. \end{cases} \tag{25}$$

Lemma 7. *Let* u_h *as in (25). Under the assumptions of Theorem 2, we have*

$$\frac{1}{2}\|u_h(t)\|^2_{L^\infty(0,T;L^2(\Omega))} + \frac{1}{p2^{p+1}}\|\nabla u_h\|^p_{L^p(0,T;L^p(\Omega))} \le \frac{1}{2}\|u_0\|^2_{L^2(\Omega)} + C\int_0^T \|f(t)\|^{p'}_{L^{p'}(\Omega)}\mathrm{d}t.$$

Proof. Choosing u_k as a test function in the weak formulation of (2), we get

$$\begin{aligned} \int_\Omega \frac{u_k^2}{h}\mathrm{d}x + \int_\Omega \Phi(\nabla u_k - \theta(u_k)) \cdot \nabla u_k \mathrm{d}x + \int_\Omega \alpha(u_k)u_k \mathrm{d}x \\ = \int_\Omega [f]_h(k-1)u_k \mathrm{d}x + \int_\Omega \frac{u_{k-1}u_k}{h}\mathrm{d}x. \end{aligned} \tag{26}$$

Since

$$u_{k-1}u_k \le \frac{u_{k-1}^2 + u_k^2}{2},$$

then (26) leads to

$$\begin{aligned} \frac{1}{2}\int_\Omega \frac{u_k^2}{h}\mathrm{d}x + \int_\Omega \Phi(\nabla u_k - \theta(u_k)) \cdot \nabla u_k \mathrm{d}x - \int_\Omega [f]_h(k-1)u_k \mathrm{d}x + \int_\Omega \alpha(u_k)u_k \mathrm{d}x \\ \le \frac{1}{2}\int_\Omega \frac{u_{k-1}^2}{h}\mathrm{d}x. \end{aligned} \tag{27}$$

Note that, by Lemma 1,

$$\begin{aligned}\Phi(\nabla u_k-\theta(u_k))\cdot\nabla u_k &= |\nabla u_k-\theta\left(u_k\right)|^{p-2}(\nabla u_k-\theta\left(u_k\right))\cdot\nabla u_k\\ &= |\nabla u_k-\theta\left(u_k\right)|^{p-2}(\nabla u_k-\theta\left(u_k\right))\cdot(\nabla u_k-\theta(u_k)+\theta(u_k))\\ &\geq \frac{1}{p}|\nabla u_k-\theta\left(u_k\right)|^{p}-\frac{1}{p}|\theta\left(u_k\right)|^{p}.\end{aligned}$$

It follows that

$$\begin{aligned}&\frac{1}{2}\int_\Omega \frac{u_k^2}{h}\mathrm{d}x+\frac{1}{p}\int_\Omega|\nabla u_k-\theta(u_k)|^p\mathrm{d}x-\int_\Omega[f]_h\left(k-1\right)u_k\mathrm{d}x-\frac{1}{p}\int_\Omega|\theta(u_k)|^p\mathrm{d}x+\int_\Omega\alpha(u_k)u_k\mathrm{d}x\\ &\quad\leq\frac{1}{2}\int_\Omega\frac{u_{k-1}^2}{h}\mathrm{d}x.\end{aligned}$$

Using Lemma (5), hypothesis (H_1), (H_2) and (3), we get

$$\frac{1}{2}\int_\Omega \frac{u_k^2}{h}\mathrm{d}x+\frac{1}{p2^p}\int_\Omega|\nabla u_k|^p\mathrm{d}x-C\left\|[f]_h\left(k-1\right)\right\|_{L^{p'}(\Omega)}^{p'}-\frac{1}{p2^{p+1}}\int_\Omega|\nabla u_k|^p\mathrm{d}x\leq\frac{1}{2}\int_\Omega\frac{u_{k-1}^2}{h}\mathrm{d}x.$$

Hence

$$\frac{1}{2}\int_\Omega \frac{u_k^2}{h}\mathrm{d}x+\frac{1}{p2^{p+1}}\int_\Omega|\nabla u_k|^p\mathrm{d}x\leq C\left\|[f]_h\left(k-1\right)\right\|_{L^{p'}(\Omega)}^{p'}+\frac{1}{2}\int_\Omega\frac{u_{k-1}^2}{h}\mathrm{d}x. \tag{28}$$

Note that for each $t\in\,]0,T]$, there exists $j\in\{0,\dots,n\}$ such that $t\in](j-1)h,jh]$. Therefore by adding the inequality (28) from $k=1$ to $k=j$, we obtain

$$\frac{1}{2}\int_\Omega u_j^2\mathrm{d}x+\frac{h}{p2^{p+1}}\sum_{k=1}^{j}\int_\Omega|\nabla u_k|^p\mathrm{d}x\leq hC\sum_{i=1}^{n}\left\|[f]_h\left(k-1\right)\right\|_{L^{p'}(\Omega)}^{p'}+\frac{1}{2}\int_\Omega u_0^2\mathrm{d}x. \tag{29}$$

Using the expression of u_h as in (25), we get

$$\frac{1}{2}\left\|u_h(t)\right\|_{L^2(\Omega)}^2+\frac{1}{p2^{p+1}}\int_0^t\int_\Omega|\nabla u_h(t)|^p\mathrm{d}x\mathrm{d}t\leq\frac{1}{2}\left\|u_0\right\|_{L^2(\Omega)}^2+C\int_0^t\left\|f(t)\right\|_{L^{p'}(\Omega)}^{p'}\mathrm{d}t.$$

Thus

$$\frac{1}{2}\left\|u_h(t)\right\|_{L^2(\Omega)}^2+\frac{1}{p2^{p+1}}\left\|\nabla u_h\right\|_{L^p(0,T;L^p(\Omega))}^p\leq\frac{1}{2}\left\|u_0\right\|_{L^2(\Omega)}^2+C\int_0^T\left\|f(t)\right\|_{L^{p'}(\Omega)}^{p'}\,dt.$$

□

Now we are ready to prove Theorem 2.

Proof. By Corollary, for $k=1,\dots,n$, there is a unique solution u_k for (2). By using Lemma 7 we may choose a subsequence u_h such that

$$u_h\rightharpoonup u \text{ weakly * in } L^\infty\left(0,T;L^2(\Omega)\right), \tag{30}$$

$$u_h\rightarrow u \text{ strongly in } L^p\left(0,T;L^p\left(\Omega\right)\right), \tag{31}$$

$$\nabla u_h\rightharpoonup\nabla u \text{ weakly in } L^p(Q), \tag{32}$$

$$\Phi(\nabla u_h-\theta(u_h))\rightharpoonup\xi \text{ weakly in } L^{p'}\left(0,T;L^{p'}\left(\Omega\right)\right). \tag{33}$$

Next, we prove that u is a weak solution of the problem (1). Let $\varphi\in C^1(\bar{Q})$ with $\varphi(.,T)=0$ and $\varphi(x,t)_\Gamma=0$. By taking $\varphi(x,kh)$ as test function, for each $k\in\{1,\dots,n\}$, we have

$$\int_\Omega\frac{u_k-u_{k-1}}{h}\varphi\mathrm{d}x+\int_\Omega\Phi(\nabla u_k-\theta(u_k))\cdot\nabla\varphi\mathrm{d}x+\int_\Omega\alpha(u_k)\varphi\mathrm{d}x=\int_\Omega[f]_h\left(\left(k-1\right)h\right)\varphi(x,kh)\mathrm{d}x.$$

Summing up all the above equalities, we have

$$\sum_{k=0}^{n-1}\int_\Omega u_k\left(\varphi(x,kh)-\varphi(x,(k+1)h)\right)\mathrm{d}x-\int_\Omega u_0\varphi(x,0)\mathrm{d}x+h\sum_{k=1}^{n}\int_\Omega \alpha(u_k)\varphi(x,kh)\mathrm{d}x$$
$$+h\sum_{k=1}^{n}\int_\Omega \Phi(\nabla u_k-\theta(u_k))\cdot\nabla\varphi(x,kh)\mathrm{d}x$$
$$=h\sum_{k=1}^{n}\int_\Omega [f]_h((k-1)h)\varphi(x,kh)\mathrm{d}x. \tag{34}$$

Since

$$\begin{aligned}\sum_{k=0}^{n-1}\int_\Omega u_k(x)[\varphi(x,kh)-\varphi\left(x,(k+1)\,h\right)]\mathrm{d}x &= -\sum_{k=0}^{n-1}\int_\Omega u_k(x)\left[\int_{kh}^{(k+1)h}\frac{\partial\varphi(x,t)}{\partial t}\right]dt\mathrm{d}x\\ &= -\sum_{k=0}^{n-1}\int_{kh}^{(k+1)h}\int_\Omega u_h(x,t)\frac{\partial\varphi(x,t)}{\partial t}\mathrm{d}x\mathrm{d}t\\ &= -\int_0^T\int_\Omega u_h(x,t)\frac{\partial\varphi(x,t)}{\partial t}\mathrm{d}x\mathrm{d}t\\ &\to -\int_0^T\int_\Omega u(x,t)\frac{\partial\varphi(x,t)}{\partial t}\mathrm{d}x\mathrm{d}t \text{ as } h\to 0,\end{aligned} \tag{35}$$

$$\begin{aligned}&h\sum_{k=1}^{n}\int_\Omega \Phi(\nabla u_h-\theta(u_h))(x,kh)\cdot\nabla\varphi(x,kh)\mathrm{d}x\\ &=\int_0^T\int_\Omega \Phi(\nabla u_h-\theta(u_h))\,(x,t)\cdot\nabla\varphi\,(x,t)\,\mathrm{d}x\mathrm{d}t\\ &\quad+\sum_{k=1}^{n}\int_{(k-1)h}^{kh}\int_\Omega \Phi(\nabla u_h-\theta(u_h)\,(x,t)\cdot\left(\nabla\varphi\,(x,kh)-\nabla\varphi\,(x,t)\right)\mathrm{d}x\mathrm{d}t\\ &\to\int_0^T\int_\Omega \xi\cdot\nabla\varphi(x,t)\mathrm{d}x\mathrm{d}t \text{ as } h\to 0,\end{aligned} \tag{36}$$

$$\begin{aligned}h\sum_{k=1}^{n}\int_\Omega [f]_h\left(x,(k-1)\,h\right)\varphi\left(x,kh\right)\mathrm{d}x &= \sum_{k=1}^{n}\int_{(k-1)h}^{kh}\int_\Omega f\left(x,\right)\varphi\left(x,kh\right)\mathrm{d}x\mathrm{d}t\\ &\to\int_0^T\int_\Omega f\varphi\mathrm{d}x\mathrm{d}t \text{ as } h\to 0,\end{aligned} \tag{37}$$

$$\begin{aligned}&h\sum_{k=1}^{n}\int_\Omega \alpha\left(u_k\right)\varphi\left(x,kh\right)\mathrm{d}x\\ &=\int_0^T\int_\Omega \alpha\left(u_h\left(x,t\right)\right)\varphi\left(x,t\right)\mathrm{d}x\mathrm{d}t-\sum_{k=1}^{n}\int_{(k-1)h}^{kh}\int_\Omega \alpha\left(u_h\right)\left(\varphi\left(x,t\right)-\varphi\left(x,kh\right)\right)\mathrm{d}x\mathrm{d}t\\ &\to\int_0^T\int_\Omega \alpha(u)\varphi\mathrm{d}x\mathrm{d}t \quad \textit{as } h\to 0,\end{aligned} \tag{38}$$

then all these relations, by letting $h \to 0$ in (34), yield

$$-\int_0^T\int_\Omega u\frac{\partial\varphi}{\partial t}\mathrm{d}x\mathrm{d}t-\int_\Omega u_0(x)\varphi(x,0)\mathrm{d}x+\int_0^T\int_\Omega \alpha(u)\varphi\mathrm{d}x\mathrm{d}t+\int_0^T\int_\Omega \xi\cdot\nabla\varphi\mathrm{d}x\mathrm{d}t$$
$$=\int_0^T\int_\Omega f\varphi\mathrm{d}x\mathrm{d}t. \tag{39}$$

If we choose $\varphi \in C_c^\infty(Q)$, we get

$$-\int_0^T\int_\Omega u\frac{\partial\varphi}{\partial t}\mathrm{d}x\mathrm{d}t+\int_0^T\int_\Omega \alpha(u)\varphi\mathrm{d}x\mathrm{d}t+\int_0^T\int_\Omega \xi\cdot\nabla\varphi\mathrm{d}x\mathrm{d}t=\int_0^T\int_\Omega f\varphi\mathrm{d}x\mathrm{d}t. \tag{40}$$

This implies that $\frac{\partial u}{\partial t}\in L^{p'}\left(0,T;W^{-1,p'}(\Omega)\right)$.
Next, we prove $\xi=\Phi(\nabla u-\theta(u))$ a.e. in Q. Let $v\in L^p\left(0,T;W_0^{1,p}(\Omega)\right)\cap L^\infty\left(0,T;L^2(\Omega)\right)$.
First we start showing that

$$\lim_{h\to 0}\int_0^T\int_\Omega(\Phi(\nabla v-\theta(u_h))\cdot(\nabla u_h-\nabla v)\mathrm{d}x\mathrm{d}t=\int_0^T\int_\Omega(\Phi(\nabla v-\theta(u))\cdot(\nabla u-\nabla v)\mathrm{d}x\mathrm{d}t. \tag{41}$$

Set

$$\int_0^T\int_\Omega(\Phi(\nabla v-\theta(u_h))\cdot(\nabla u_h-\nabla v)\,\mathrm{d}x\mathrm{d}t=A_h+B_h, \tag{42}$$

where

$$A_h=\int_0^T\int_\Omega(\Phi(\nabla v-\theta(u_h))-(\Phi(\nabla v-\theta(u))\cdot(\nabla u_h-\nabla v)\mathrm{d}x\mathrm{d}t,$$
$$B_h=\int_0^T\int_\Omega(\Phi(\nabla v-\theta(u))\cdot(\nabla u_h-\nabla v)\mathrm{d}x\mathrm{d}t.$$

By applying again Hölder's inequality, we get

$$|A_h|\le\|\Phi(\nabla v-\theta(u_h))-\Phi(\nabla v-\theta(u))\|_{L^{p'}(Q)}\|\nabla u_h-\nabla v\|_{L^p(Q)}.$$

By hypothesis (H_2), we have

$$\int_0^T\int_\Omega|\theta(u_h)-\theta(u)|^p\mathrm{d}x\mathrm{d}t\le\lambda_2^p\int_0^T\int_\Omega|u_h-u|^p\mathrm{d}x\mathrm{d}t. \tag{43}$$

In (31), $\theta(u_h)\to\theta(u)$ strongly in $L^p(Q)$, which implies that $\Phi(v-\theta(u_h))\to\Phi(v-\theta(u))$ strongly in $L^{p'}(Q)$. Using (32), we obtain

$$\lim_{h\to 0}A_h=0. \tag{44}$$

On the other hand

$$\lim_{h\to 0}B_h=\int_0^T\int_\Omega(\Phi(\nabla v-\theta(u))\cdot(\nabla u-\nabla v)\mathrm{d}x\mathrm{d}t. \tag{45}$$

Hence

$$\lim_{h\to 0}\int_0^T\int_\Omega(\Phi(\nabla v-\theta(u_h))\cdot(\nabla u_h-\nabla v)\mathrm{d}x\mathrm{d}t=\int_0^T\int_\Omega(\Phi(\nabla v-\theta(u))\cdot(\nabla u-\nabla v)\mathrm{d}x\mathrm{d}t. \tag{46}$$

Now summing up the above inequalities (27), for $k = 1, \ldots, n$, we have

$$\begin{aligned}&\frac{1}{2}\int_\Omega u_h^2(T)\mathrm{d}x + \int_0^T\int_\Omega \Phi(\nabla u_h - \theta(u_h)) \cdot \nabla u_h \mathrm{d}x\mathrm{d}t + \int_0^T\int_\Omega \alpha(u_h)u_h \mathrm{d}x\mathrm{d}t \\ &\leq \int_0^T\int_\Omega f u_h \mathrm{d}x\mathrm{d}t + \frac{1}{2}\int_\Omega u_0^2 \mathrm{d}x. \end{aligned} \tag{47}$$

Since

$$\begin{aligned}\Phi(\nabla u_h - \theta(u_h)) \cdot \nabla u_h = &[\Phi(\nabla u_h - \theta(u_h)) - \Phi(\nabla v - \theta(u_h))] \cdot (\nabla u_h - \nabla v) \\ &+ \Phi(\nabla u_h - \theta(u_h)) \cdot \nabla v + \Phi(\nabla v - \theta(u_h)) \cdot (\nabla u_h - \nabla v)\,,\end{aligned}$$

then from Lemma 3, we obtain

$$\Phi(\nabla u_h - \theta(u_h)) \cdot \nabla u_h \geq \Phi(\nabla u_h - \theta(u_h)) \cdot \nabla v + \Phi(\nabla v - \theta(u_h)) \cdot (\nabla u_h - \nabla v)\,.$$

This implies, by using (47), that

$$\begin{aligned}\frac{1}{2}\int_\Omega u_0^2 \mathrm{d}x + \int_0^T\int_\Omega f u_h \mathrm{d}x\mathrm{d}t \geq &\frac{1}{2}\int_\Omega u_h^2(T)\mathrm{d}x + \int_0^T\int_\Omega (\Phi(\nabla u_h - \theta(u_h)) \cdot \nabla v \mathrm{d}x\mathrm{d}t \\ &+ \int_0^T\int_\Omega (\Phi(\nabla v - \theta(u_h)) \cdot (\nabla u_h - \nabla v)\mathrm{d}x\mathrm{d}t \\ &+ \int_0^T\int_\Omega \alpha(u_h)u_h \mathrm{d}x\mathrm{d}t.\end{aligned}$$

Passing $h \to 0$, we have

$$\begin{aligned}\frac{1}{2}\int_\Omega u_0^2 \mathrm{d}x + \int_0^T\int_\Omega f u \mathrm{d}x\mathrm{d}t \geq &\frac{1}{2}\int_\Omega u^2(T)\mathrm{d}x + \int_0^T\int_\Omega \xi \cdot \nabla v \mathrm{d}x\mathrm{d}t \\ &+ \int_0^T\int_\Omega (\Phi(\nabla v - \theta(u)) \cdot (\nabla u - \nabla v)\mathrm{d}x\mathrm{d}t \\ &+ \int_0^T\int_\Omega \alpha(u)u \mathrm{d}x\mathrm{d}t.\end{aligned} \tag{48}$$

Now if we choose u as test function in (39), we get

$$-\frac{1}{2}\int_\Omega u^2(T)\mathrm{d}x + \int_0^T\int_\Omega \alpha(u)u \mathrm{d}x\mathrm{d}t + \int_0^T\int_\Omega \xi \cdot \nabla u \mathrm{d}x\mathrm{d}t = \int_0^T\int_\Omega f u \mathrm{d}x\mathrm{d}t + \frac{1}{2}\int_\Omega u_0^2 \mathrm{d}x. \tag{49}$$

Combining (48) with (49), we obtain

$$\int_0^T\int_\Omega (\xi - (\Phi(\nabla v - \theta(u))) \cdot (\nabla v - \nabla u)\mathrm{d}x\mathrm{d}t \leq 0.$$

Choosing $v = u - \lambda\Psi$, where $\lambda > 0$ and $\Psi \in L^p\left(0, T; W_0^{1,p}(\Omega)\right) \cap L^\infty\left(0, T; L^2(\Omega)\right)$, in the above inequality to have

$$\int_0^T\int_\Omega (\xi - \Phi\left(u - \lambda\Psi - \theta(u)\right)) \cdot \nabla\Psi \mathrm{d}x\mathrm{d}t \geq 0.$$

Passing the limits as $\lambda \to 0^+$ and using Lebesgue's dominated convergence Theorem, we deduce

$$\int_0^T\int_\Omega (\xi - \Phi(u - \theta(u)) \cdot \psi \mathrm{d}x\mathrm{d}\tau \geq 0, \quad \forall \psi \in \left(L^p\left(0, T; W_0^{1,p}(\Omega)\right) \,\cap\, L^\infty\left(0, T; L^2(\Omega)\right)\right)^N.$$

This implies that $\xi = \Phi(\nabla u - \theta(u))$ a.e in Q. Hence, by (40), we conclude that

$$-\int_0^T \int_\Omega u \frac{\partial \varphi}{\partial t} \mathrm{d}x\mathrm{d}t + \int_0^T \int_\Omega \alpha(u)\varphi \mathrm{d}x\mathrm{d}t + \int_0^T \int_\Omega \Phi\left(\nabla u - \theta\left(u\right)\right) \cdot \nabla\varphi \mathrm{d}x\mathrm{d}t = \int_0^T \int_\Omega f\varphi \mathrm{d}x\mathrm{d}t.$$

Recalling the fact that $u \in L^p\left(0, T; W_0^{1,p}(\Omega)\right) \cap L^\infty\left(0, T; L^2(\Omega)\right)$ and $\frac{\partial u}{\partial t} \in L^{p'}\left(0, T\,; W^{-1,p'}(\Omega)\right)$ from the equation, we conclude that u belongs to $C\left(0, T; L^2(\Omega)\right)$, which completes our proof of the existence of the weak solutions of the problem (1).
The proof of the uniqueness of the solution of the equation (1) is similar to the proof of Theorem 3.3 in El Hachimi et al. (2010) and so it is omitted.

□

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